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Exact Distance Oracles for Planar Graphs with Failing Vertices

by
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Abstract

We consider exact distance oracles for directed weighted planar graphs in the presence of failing vertices. Given a source vertex u, a target vertex v and a set X of k failed vertices, such an oracle returns the length of a shortest u-to-v path that avoids all vertices in X. We propose oracles that can handle any number k of failures. More specifically, for a directed weighted planar graph with n vertices, any constant k, and for any $q \in [1, \sqrt{n}]$, we propose an oracle of size $\tilde{\mathcal{O}}(\frac{n^{k+3/2}}{q^{2k+1}})$ that answers queries in $\tilde{\mathcal{O}}(q)$ time.[1] In particular, we show an $\tilde{\mathcal{O}}(n)$-size, $\tilde{\mathcal{O}}(\sqrt{n})$-query-time oracle for any constant k. This matches, up to polylogarithmic factors, the fastest failure-free distance oracles with nearly linear space. For single vertex failures ($k = 1$), our $\tilde{\mathcal{O}}(\frac{n^{5/2}}{q^3})$-size, $\tilde{\mathcal{O}}(q)$-query-time oracle improves over the previously best known tradeoff of Baswana et al. [SODA 2012] by polynomial factors for $q = \Omega(n^t)$, $t \in (1/4, 1/2]$. For multiple failures, no planarity exploiting results were previously known.

[1] The $\tilde{\mathcal{O}}(\cdot)$ notation hides polylogarithmic factors.

Contents

1 Introduction

The contents of this section were partially extracted from [41].

Distance Oracles Computing shortest paths is one of the most well-studied algorithmic problems. The shortest-path query problem is different from the classical *single-source* (SSSP) and *all-pairs shortest paths* (APSP) problems in that there are two stages: *preprocessing* and answering *queries*. We are first presented with a *network* (also termed *graph*). A so-called *preprocessing* algorithm may compute certain information (a *data structure* or *index*, in the theory community referred to as a *distance oracle* [43]) to prepare for the second phase. After this preprocessing step, applications may ask *queries*, which should be answered efficiently.

A lazy solution to the shortest-path query problem is not to precompute any data structure at all but to use an SSSP algorithm [12, 22] to answer queries. Answering a query then requires time roughly linear in the network size. An eager solution is to precompute the results for all possible queries using an APSP algorithm [20, 44, 29]. Both solutions have their advantages and disadvantages: for the SSSP strategy, no preprocessing is necessary but the query processing is rather slow; for the APSP strategy, the query execution is extremely fast: one table lookup suffices to obtain the shortest-path distance; but the preprocessing step is expensive and the space consumption is prohibitively large for many real-world networks, spanning millions or even billions of nodes. In the shortest-path query scenario, we mediate between these two extremes, that is, we analyze the *tradeoff* between space, preprocessing time, and query time. If the query algorithm is allowed to return an approximate shortest path, the worst-case *accuracy* (often called *stretch*) is also an important factor of the tradeoff. Designing a shortest-path query processing method raises questions such as: How can these data structures be computed efficiently? What amount of storage space is necessary? How much improvement of the query time is possible? How good is the approximation quality of the query result? What are the tradeoffs between pre-computation time, space, query time, and approximation quality?

Distance Oracles for Planar Graphs Efficiently finding "good" routes in transportation networks is arguably the main application scenario for shortest-path query methods. Due to the importance of planar graphs as a more-or-less accurate model for road networks, shortest-path queries for planar graphs have been studied extensively over the past three decades [13, 3, 9, 18, 37, 6, 10, 25].

Exact distance Oracles for Planar Graphs Djidjev [13], improving upon [19], proves that, for any $S \in [n, n^2]$, there is an exact distance oracle using space $O(S)$ with query time $O(n^2/S)$. The oracle uses two common main concepts, r-divisions and *portals*. An r-division is a partition of the edges into $O(n/r)$ *regions* of size $O(r)$ such that each region R has at most $O(\sqrt{r})$ *boundary nodes* ∂R (a node is called a boundary node if it is adjacent to edges in different regions). *Portals*, are a set of carefully selected points (usually a subset of the node set of the graph) which represent shortest paths. The oracle defines a set of $O(n/\sqrt{r})$ *portals* to be the set of boundary nodes on the r-division. Pairwise distances among all portals are computed and stored. The space requirements for this distance oracle are $S = O((n/\sqrt{r})^2) = O(n^2/r)$.

At query time, a pair of nodes (s, t) is given. Let R_s and R_t be the regions that contain s and t respectively in the r-division. The algorithm first searches (using SSSP [26]) both regions R_s and R_t. If s and t are in the same region R and if the shortest path is entirely contained in R, the shortest-path distance has been found in time $O(r)$ (short-range query). Otherwise, exact distances to all corresponding portals (boundary nodes in $\partial R_s, \partial R_t$, respectively) have been computed, and the distance is the minimum among all pairs of portals $(p_s, p_t) \in \partial R_s \times \partial R_t$ of $d(s, p_s) + d(p_s, p_t) + d(p_t, t)$ where $d(.,.)$ is the shortest distance between two nodes. Since the number of boundary node pairs is bounded by $O((\sqrt{r})^2)$, the query time for long-range queries is also $O(r)$.

Fakcharoenphol and Rao [18] exploit a certain property due to the planarity of the graph which is known as the Monge property [36] (a definition will be given in Section 2). They

represent the distances among the boundary vertices of each piece in a complete bipartite (non-planar) graph, which they call a Dense distance graph. Their query algorithm can run Dijkstra's algorithm on a union of DDGs in time roughly proportional only to the number of nodes in these DDGs (as opposed to the number of edges, which for DDGs is quadratic in the number of nodes). They obtain a distance oracle of size $\tilde{\mathcal{O}}(n)$, that answer queries in $\tilde{\mathcal{O}}(\sqrt{n})$ time. This technique is used for various distance oracles with low space and preprocessing complexities [18, 28, 40, 37].

The known space to query-time tradeoffs have been significantly improved very recently [25, 10]. The currently best known tradeoff for exact distance oracles in planar graph is an oracle of size $\tilde{\mathcal{O}}(n^{3/2}/q)$, that answers queries in time $\tilde{\mathcal{O}}(q)$ for any $q \in [1, n^{1/2}]$ [25]. One of the main ideas is the use of *Voronoi diagram* which was previously used in [10]. More details about this algorithm will be given in Section 2. Note that all known oracles with nearly linear (i.e. $\tilde{\mathcal{O}}(n)$) space require $\Omega(\sqrt{n})$ query time.

Approximate distance oracles for Planar Graphs To obtain constant or polylogarithmic query times while maintaining almost linear space, approximate distance oracles were considered. Thorup [42] presents efficient $(1+\epsilon)$-approximate distance oracles for directed planar graphs. One of the main ingredients of Thorup's construction is a special separator consisting of a constant number of shortest paths. In contrast to the standard small $O(\sqrt{n})$ separator, a shortest path separator might contain $\Omega(n)$ nodes. The main benefit of such separators is the fact that shortest paths between two nodes in a directed planar graph, cross a shortest path separator at most once. Each node computes and stores shortest-path distances to a set of $O(1/\epsilon)$ portals per level, recursively for $O(\log n)$ levels. It yields a near linear space approximate distance oracle with $\tilde{\mathcal{O}}(1)$ query time for a weighted planar digraph.

Dynamic distance oracles In recent decades researchers have investigated the shortest path problem in graphs subject to failures, or more broadly, to changes. One such variant is the *replacement paths problem*. In this problem we are given a graph G and vertices u and

v. The goal is to report the u-to-v distance in G for each possible failure of a single edge along the shortest u-to-v path. Another variant is that of constructing a distance oracle that answers u-to-v distance queries subject to edge or vertex failures (u, v and the set of failures are given at query time). Perhaps the most general of these variants is the *fully-dynamic* distance oracle; a data structure that supports distance queries as well as updates to the graph such as changes to edge lengths, edge insertions or deletions and vertex insertions or deletions.

One obvious but important application of handling failures is in geographical routing. Further motivation for studying this problem originates from Vickrey pricing in networks [39, 27]; see [11] for a concise discussion on the relation between the problems. A long-studied generalization of the shortest path problem is the the k-shortest path, in which not one but several shortest paths must be produced between a pair of vertices. This problem reduces to running k executions of a replacement paths algorithm, and has many applications itself [16].

Demetrescu et al. presented an $\mathcal{O}(n^2 \log n)$-size oracle answering single failure distance queries in constant time [11]. Bernstein and Karger, improved the construction time in [5]. Interestingly, Duan and Pettie, building upon this work, showed an $\mathcal{O}(n^2 \log^3 n)$-size oracle that can report distances subject to two failures, in time $\mathcal{O}(\log n)$ [14]. Based on this oracle, they then easily obtain an $\tilde{\mathcal{O}}(n^k)$-space oracle answering distance queries in $\tilde{\mathcal{O}}(1)$ time for any $k \geq 2$. Oracles that require less space for more than 2 failures have been proposed, such as the one presented in [45], but at the expense of $\Omega(n)$ query time. Such oracles are unsatisfactory for planar graphs, where single source shortest paths can be computed in linear or nearly linear time.

For planar graphs, the replacement paths problem (i.e. when both the source and destination are fixed in advance) can be solved in nearly linear time [15, 33, 46].

For the single source, single failure version of the problem (i.e. when the source vertex is fixed at construction time, and the query specifies just the target and a single failed vertex), Baswana et al. [4] presented an oracle with size and construction time $\mathcal{O}(n \log^4 n)$

that answers queries in $\mathcal{O}(\log^3 n)$ time. They then showed an oracle of size $\tilde{\mathcal{O}}(n^2/q)$ for the general single failure problem (i.e. when the source, destination, and failed vertex are all specified at query time), that answers queries in time $\tilde{\mathcal{O}}(q)$ for any $q \in [1, n^{1/2}]$. They conclude the paper by asking whether it is possible to design a compact distance oracle for a planar digraph which can handle multiple vertex failures. We answer this question in the affirmative.

Fakcharoenphol and Rao, in their seminal paper [18], presented distance oracles that require $\mathcal{O}(n^{2/3} \log^{7/3} n)$ and $\mathcal{O}(n^{4/5} \log^{13/5} n)$ amortized time per update and query for non-negative and arbitrary edge-weight updates respectively.[2] The space required by these oracles is $\mathcal{O}(n \log n)$. Klein presented a similar data structure in [31] for the case where edge-weight updates are non-negative, requiring time $\mathcal{O}(n^{2/3} \log^{5/3} n)$. Klein's result was extended in [28], where, assuming non-negativity of edge-weight updates, the authors showed how to handle edge deletions and insertions (not violating the planarity of the embedding), and in [30], where the authors showed how to handle negative edge-weight updates, all within the same time complexity. In fact, these results can all be combined, and along with a recent slight improvement on the running time of FR-Dijkstra [24], they yield a dynamic distance oracle that can handle any of the aforementioned edge updates and queries within time $\mathcal{O}(n^{2/3} \frac{\log^{5/3} n}{\log^{4/3} \log n})$. We further extend these results by showing that vertex deletions and insertions can also be handled within the same time complexity. The main challenge lies in handling vertices of high degree.

On the lower bounds side, it is known that an exact dynamic oracle requiring amortized time $\mathcal{O}(n^{1/2-\delta})$, for any constant $\delta > 0$, for both edge-weight updates and distance queries, would refute the APSP conjecture, i.e. that there is no truly subcubic combinatorial algorithm for solving the all-pairs shortest path problems in weighted (general) graphs [1].

Abraham et al. [2] gave a $(1+\epsilon)$ labeling scheme for undirected planar graphs with polylogarithmic size labels, such that a $(1+\epsilon)$-approximation of the distance between vertices

[2]Though this is not mentioned in [18], the query time can be made worst case rather than amortized by standard techniques.

u and v in the presence of $|F|$ vertex or edge failures can be recovered from the labels of u, v and the labels of the failed vertices in $\tilde{\mathcal{O}}(|F|^2)$ time. They then use this labeling scheme to devise a fully dynamic $(1+\epsilon)$-distance oracle with size $\tilde{\mathcal{O}}(n)$ and $\tilde{\mathcal{O}}(\sqrt{n})$ query and update time.[3] See Appendix A for more details about this result.

In this thesis we focus on these problems, and in particular on handling vertex failures in planar graphs. Observe that edge failures easily reduce to vertex failures. Indeed, by replacing each edge (a, c) of G with a new dummy vertex b and appropriately weighted edges (a, b) and (b, c); the failure of edge (a, c) in G corresponds to the failure of vertex b in the new graph. Note that this transformation does not depend on planarity. In sparse graphs, such as planar graphs, this transformation only increases the number of vertices by a constant factor. Also note that there is no such obvious reduction in the other direction that preserves planarity. In general graphs, one can replace each vertex v by two vertices v_{in} and v_{out}, assign to v_{in} (resp. v_{out}) all the edges incoming to v (resp. outgoing from v) and add a 0-length directed edge e from v_{in} to v_{out}. The failure of vertex v in the original graph corresponds to the failure of edge e in the new graph. However, this transformation does not preserve planarity.

1.1 Our Results and Techniques

In this work we focus on distance queries subject to vertex failures in planar graphs. Our results can be summarized as follows.

1. We show how to preprocess a directed weighted planar graph G in $\tilde{\mathcal{O}}(n)$ time into an oracle of size $\tilde{\mathcal{O}}(n)$ that, given a source vertex u, a target vertex v, and a set X of k failing vertices, reports the length of a shortest u-to-v path in $G \setminus X$ in $\tilde{\mathcal{O}}(\sqrt{kn})$ time. See Lemma 12.

2. For k allowed failures, and for any $r \in [1, n]$, we show how to construct an $\tilde{\mathcal{O}}(\frac{n^{k+1}}{r^{k+1}}\sqrt{nr})$-

[3] A fully dynamic distance oracle supports arbitrary edge and vertex insertions and deletions, and length updates.

size oracle that answers queries in time $\tilde{\mathcal{O}}(k\sqrt{r})$. See Theorem 15. For $k = 1$, this improves over the previously best known tradeoff of Baswana et al. [4] by polynomial factors for $r = \Omega(n^t)$, $t \in (1/2, 1]$. To the best of our knowledge, this is the first tradeoff for $k > 1$. See Fig. 1.

3. We extend the exact dynamic distance oracles mentioned in the previous section to also handle vertex insertions and deletions without changing their space and time bounds.

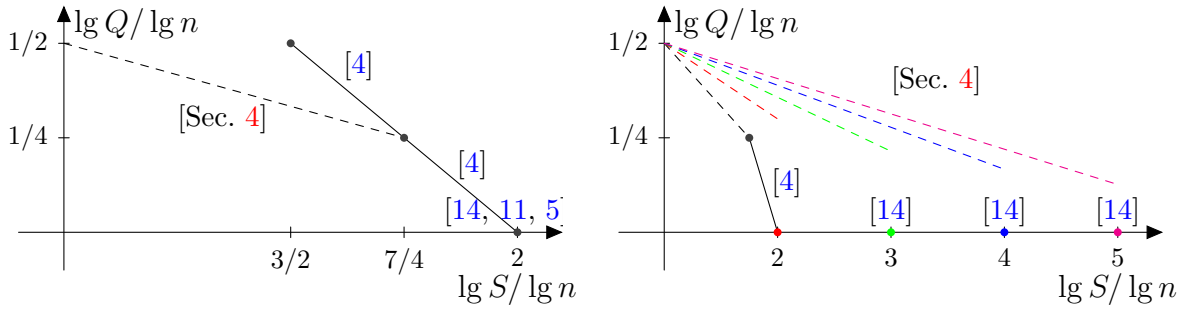


Figure 1: Left: Tradeoff of the Space (S) vs. the Query time (Q) for exact distance oracles for a single failed vertex (i.e. $k = 1$) on a doubly logarithmic scale, ignoring constant and logarithmic factors. The previous tradeoff is indicated by a solid line, while the new tradeoff is indicated by a dashed line. Right: the same tradeoff for $k = 1, \ldots, 5$, shown with different colours. The points on the x-axis correspond to the result of [14], while the new tradeoffs are indicated by dashed lines.

Our nearly-linear space oracle that reports distances in the presence of k failures in $\tilde{\mathcal{O}}(\sqrt{kn})$ time is obtained by adapting a technique of Fakcharoenphol and Rao [18]. In a nutshell, a planar graph can be recursively decomposed using small cycle separators, such that the boundary of each piece in the decomposition (i.e. the vertices of a piece that also belong to other pieces in the decomposition) is a cycle with relatively few vertices. Instead of working with the given planar graph, one computes distances over its *dense distance graph* (DDG); a non-planar graph on the boundary vertices of the pieces which captures the distances between boundary vertices within each of the underlying pieces. Fakcharoenphol and Rao developed an efficient implementation of Dijkstra's algorithm on the DDG. This algorithm, nicknamed *FR-Dijkstra*, runs in time roughly proportional to the number of vertices of the

DDG (i.e. boundary vertices), rather than in time proportional to the number of vertices in the planar graph. Roughly speaking, Fakcharoenphol and Rao show that to obtain distances from u to v with k edge failures, it (roughly) suffices to consider just the boundary vertices of the pieces in the recursive decomposition that contain failed edges. Since pieces at the same level of the recursive decomposition are edge-disjoint, the total number of boundary vertices in all the required pieces is only $\mathcal{O}(\sqrt{kn})$. This $\tilde{\mathcal{O}}(n)$-size, $\tilde{\mathcal{O}}(\sqrt{kn})$-query-time oracle, supporting distance queries subject to a batch of k edge cost updates, leads to their dynamic distance oracle.

The difficulty in handling vertex failures is that a high degree vertex x may be a boundary vertex of many (possibly $\Omega(n)$) pieces in the recursive decomposition. Then, if x fails, one would have to consider too many pieces and too many boundary vertices. Standard techniques such as degree reduction by vertex splitting are inappropriate because when a vertex fails all its copies fail. To overcome this difficulty we define a variant of the dense distance graph which, instead of capturing shortest path distances between boundary vertices within a piece, only captures distances of paths that are internally disjoint from the boundary. We show that such distances can be computed efficiently, and that it then suffices to include in the FR-Dijkstra computation (roughly) only pieces that contain x, but not as a boundary vertex. This leads to our nearly-linear-space oracle reporting distances in the presence of k failures in $\tilde{\mathcal{O}}(\sqrt{kn})$ time (item 1 above). See Section 3. Plugging the same technique into the existing dynamic distance oracles extends them to support vertex deletions (item 3 above). See Section 6.

Our main result, the space vs. query-time tradeoff (item 2 above), is obtained by a non-trivial combination of this technique with ideas from the recent static distance oracle presented in [25]. Namely, by a combination of FR-Dijkstra on our variant of the DDG with r-divisions, external DDGs, and efficient point location in Voronoi diagrams. See Sections 4 and 5.

1.2 Road map

Lemma 12 is proved in Section 3. Theorem 15 is proved in Section 4 except for the preprocessing of the distance oracle, which is proved in Section 5. Finally, in Section 6 we describe how to achieve item 3 above.

2 Preliminaries

In this section we review the main techniques required for describing our result. Throughout the paper we consider a weighted directed planar graph $G = (V(G), E(G))$, embedded in the plane. (We use the terms weight and length for edges and paths interchangeably throughout the paper.) We use $|G|$ to denote the number of vertices in G. Since planar graphs are sparse, $|E(G)| = \mathcal{O}(|G|)$ as well. For an edge (u, v), we say that u is its tail and v is its head. $d_G(u, v)$ denotes the distance from u to v in G. We denote by $d_G(u, v, X)$ the distance from u to v in $G \setminus X$, where $X \in V(G)$ or $X \subset V(G)$. If the reference graph is clear from the context we may omit the subscript. We assume that the input graph has no negative length cycles. If it does, we can detect this in $\mathcal{O}(n\frac{\log^2 n}{\log\log n})$ time by computing single source shortest paths from any vertex [38]. In the same time complexity, we can transform the graph in a standard way so that all edge weights are non-negative and distances are preserved. We further assume that shortest paths are unique as required for a result from [23] that we use; this can be ensured in $\mathcal{O}(n)$ time by a deterministic perturbation of the edge weights [17]. Each original distance can be recovered from the corresponding distance in the transformed graph in constant time.

Separators and recursive decompositions in planar graphs. Miller [35] showed how to compute a Jordan curve that intersects the graph at $\mathcal{O}(\sqrt{n})$ nodes and separates it into two pieces with at most $2n/3$ vertices each. Jordan curve separators can be used to recursively separate a planar graph until pieces have constant size. The authors of [32] show how to

obtain a complete recursive decomposition tree $\mathcal{T}$ of G in $\mathcal{O}(n)$ time. $\mathcal{T}$ is a binary tree whose nodes correspond to subgpraphs of G (pieces), with the root being all of G and the leaves being pieces of constant size. We identify each piece P with the node representing it in $\mathcal{T}$. We can thus abuse notation and write $P \in \mathcal{T}$. An r-division [21] of a planar graph, for $r \in [1, n]$, is a decomposition of the graph into $\mathcal{O}(n/r)$ pieces, each of size $\mathcal{O}(r)$, such that each piece has $\mathcal{O}(\sqrt{r})$ boundary vertices, i.e. vertices incident to edges in other pieces. Another usually desired property of an r-division is that the boundary vertices lie on a constant number of faces of the piece (holes). For every r larger than some constant, an r-division with this property (i.e. few holes per piece) is represented in the decomposition tree $\mathcal{T}$ of [32]. Throughout the paper, to avoid confusion, we use "nodes" when referring to $\mathcal{T}$ and "vertices" when referring to G. We denote the boundary vertices of a piece P by ∂P. We refer to non-boundary vertices as internal.

Lemma 1 ([25])**.** *Each node in $\mathcal{T}$ corresponds to a piece such that (i) each piece has $\mathcal{O}(1)$ holes, (ii) the number of vertices in a piece at depth ℓ in $\mathcal{T}$ is $\mathcal{O}(n/c_1^\ell)$, for some constant $c_1 > 1$, (iii) the number of boundary vertices in a piece at depth ℓ in $\mathcal{T}$ is $\mathcal{O}(\sqrt{n}/c_2^\ell)$, for some constant $c_2 > 1$.*

We use the following well-known bounds (see e.g., [25]).

Proposition 2. $\sum_{P\in\mathcal{T}} |P| = \mathcal{O}(n \log n)$, $\sum_{P\in\mathcal{T}} |\partial P| = \mathcal{O}(n)$ *and* $\sum_{P\in\mathcal{T}} |\partial P|^2 = \mathcal{O}(n \log n)$.

We show the following bound that will be used in future proofs.

Proposition 3. $\sum_{P\in\mathcal{T}} |P||\partial P|^2 = \mathcal{O}(n^2)$.

Proof. Let $P_1^\ell, P_2^\ell, \ldots, P_j^\ell$ be the pieces at the ℓ-th level of the decomposition. $\sum_i |P_i^\ell| = \mathcal{O}(n)$ since the pieces are edge-disjoint. We know by Lemma 1 that $|\partial P_j^\ell| = \mathcal{O}(\sqrt{n}/c_2^\ell)$ for all j and hence $|\partial P_j^\ell|^2 = \mathcal{O}(n/c_2^{2\ell})$ for all j. It follows that $\sum_i |P_i^\ell||\partial P_i^\ell|^2 = \mathcal{O}(n^2/c_2^{2\ell})$ and the claimed bound follows by summing over all levels of $\mathcal{T}$. □

Dense distance graphs and FR-Dijkstra. The *dense distance graph* of a piece P, denoted DDG_P is a complete directed graph on the boundary vertices of P. Each edge (u, v) has weight $d_P(u, v)$, equal to the length of the shortest u-to-v path in P. DDG_P can be computed in time $\mathcal{O}(|\partial P|^2 + |P| \log |P|)$ using the multiple source shortest paths (MSSP) algorithm [31, 7]. Over all pieces of the recursive decomposition this takes time $\mathcal{O}(n \log^2 n)$ in total and requires $\mathcal{O}(n \log n)$ space by Proposition 2. We next give a —convenient for our purposes— interface for FR-Dijkstra [18], which is an efficient implementation of Dijkstra's algorithm on any union of DDGs. The algorithm exploits the fact that, due to planarity, certain submatrices of the adjacency matrix of DDG_P satisfy the Monge property. (A matrix M satisfies the Monge property if, for all $i < i'$ and $j < j'$, $M_{i,j} + M_{i',j'} \leq M_{i',j} + M_{i,j'}$ [36].) The interface is specified in the following theorem, which was essentially proved in [18], with some additional components and details from [30, 38].

Theorem 4 ([18, 30, 38])**.** *A set of DDGs with $\mathcal{O}(M)$ vertices in total (with multiplicities), each having at most m vertices, can be preprocessed in time and space $\mathcal{O}(M \log m)$ in total. After this preprocessing, Dijkstra's algorithm can be run on the union of any subset of these DDGs with $\mathcal{O}(N)$ vertices in total (with multiplicities) in time $\mathcal{O}(N \log^2 m)$, by relaxing edges in batches. Each such batch consists of edges that have the same tail.*

The algorithm in the above theorem is called FR-Dijkstra. It is useful in computing distances in sublinear time, as demonstrated by the following lemma and corollary.

Definition 5. *Let u be a vertex. Let P_u be a leaf piece in $\mathcal{T}$ containing u. A* cone *of u is the union of the following DDGs: (i) DDG_{P_u}, with u considered a boundary vertex of P_u. (ii) For every (not necessarily strict) ancestor P of P_u, DDG_Q of the sibling Q of P.*

Lemma 6. *Let x and y be two vertices in the cone of a vertex u. The x-to-y distance in G equals the x-to-y distance in the cone of u.*

Proof. Let $P_u = P_0, P_1, \ldots, P_d = G$ be the ancestors of P_u ordered by decreasing depth in $\mathcal{T}$. Let Q_i be the sibling of P_i in $\mathcal{T}$. Let cone_i be $DDG_{P_u} \cup \bigcup_{j<i} DDG_{Q_j}$. We will prove by

induction that for any two vertices $x, y \in \mathsf{cone}_i$, the x-to-y distance in P_i equals the x-to-y distance in cone_i. This statement is trivially true for $i = 0$. Let us assume it is true for k. Consider an x-to-y shortest path p in P_{k+1}, where $x, y \in \mathsf{cone}_{k+1}$. Path p can be decomposed into maximal subpaths that are entirely contained in P_k or Q_k and whose endpoints are in $\{x, y\} \cup (\partial P_k \cap \partial Q_k)$. For each such subpath we either have a path with the same length in cone_k by the inductive assumption, or an edge of DDG_{Q_k}. This shows that the length of p is at least the length of the x-to-y distance in cone_k. Since every edge of cone_k corresponds to some path in P_k, the opposite also holds, so the two quantities are equal. □

Corollary 7. *Let u, v be two distinct vertices in G. Let p be a shortest u-to-v path in G. If p is not fully contained in P_u then we can compute the length of p by running FR-Dijkstra on the union of the cone of u and the cone of v. This takes $O(\sqrt{n})$ time.*

Proof. Since p is not fully contained in P_u, p must visit a vertex w in the separator of the LCA of P_u and P_v in $\mathcal{T}$. We are done by decomposing p into the prefix ending at w and the suffix beginning at w an, and applying Lemma 6. The running time follows by Theorem 4 and Lemma 1. □

Voronoi diagrams with point location. In mathematics, a *Voronoi diagram* is a partitioning of a plane into regions based on distance to *sites* in a specific subset of the plane. The set of sites is specified beforehand, and for each site there is a corresponding region consisting of all points closer to that site than to any other. These regions are called *Voronoi cells*.

Let P be a directed planar graph with real edge-lengths, and no negative-length cycles. Let S be a set of vertices that lie on a single face of P; we choose the elements of S as the sites for the (graphical) Voronoi diagram of P. Each site $u \in S$ has a weight $\omega(s) \geq 0$ associated with it. The additively weighted distance between a site $s \in S$ and a vertex $v \in V$, denoted by $d^{\omega}_P(s, v)$ is defined as $\omega(s)$ plus the length of the s-to-v shortest path in P.

Definition 8. *The additively weighted Voronoi diagram of (S, ω) ($VD(S, \omega)$) within P is a partition of $V(P)$ into pairwise disjoint sets, one set $\mathsf{Vor}(s)$ for each site $s \in S$. The set*

$\mathsf{Vor}(s)$ which is called the Voronoi cell of s, contains all vertices in $V(P)$ that are closer (w.r.t. $d_P^{\omega}(.\ ,\ .)$) to s than to any other site in S (assuming that the distances are unique). There is a dual representation $VD^*(S,\omega)$ of a Voronoi diagram $VD(S,\omega)$ as a planar graph with $\mathcal{O}(|S|)$ vertices and edges.

Theorem 9 ([25, 23]). *Given subsets $S'_1,\ldots,S'_m$ of S, and additive weights $\omega_i(u)$ for each $u \in S'_i$, we can construct a data structure of size $\mathcal{O}(|P|\log|P| + \sum_i |S'_i|)$ that supports the following (*point location*) queries. Given i, and a vertex v of P, report in $\mathcal{O}(\log^2|P|)$ time the site s in the additively weighted Voronoi diagram $VD(S_i,\omega_i)$ such that v belongs to $\mathsf{Vor}(s)$ and the distance $d_P^{\omega_i}(s,v)$. The time and space required to construct this data structure are $\tilde{\mathcal{O}}(|P||S|^2 + \sum_i |S'_i|)$.*

We now describe the distance oracle of [25], for exact distance oracles in planar graph which uses $\mathcal{O}(n^{3/2})$-space and answers queries in $\mathcal{O}(\log n)$ time. This description is given as a background to the reader since we will adapt these ideas in Section 4.

First, we compute a recursive decomposition of G using Jordan separators as described above. For each piece $R = (V_R, E_R)$ in the recursive decomposition we perform the following preprocessing. We compute and store, for each boundary node v of R, the shortest path tree T_v^R in R rooted at v. Additionally, we store for every node u of R the distance from v to u and the distance from u to v in the whole G. For a non-terminal piece R, let $P = (V_P, E_P)$ and $Q = (V_Q, E_Q)$ be the two pieces into which R is separated. For every node $u \in V_Q$ and for every hole h of P we store an additively weighted Voronoi diagram $VD(S_h,\omega)$ for P, where the set of sites S_h is the set of boundary nodes of P incident to the hole h, and the additive weights ω correspond to the distances in G from u to each site in S_h. We enhance each Voronoi diagram with the point location data structure of Theorem 9. We also store the same information with the roles of Q and P exchanged.

Consider a piece R with $n(R)$ nodes and $b(R)$ boundary nodes. The trees T_v^R and the stored distances in G require a total of $O(b(R)n(R))$ space. Let R be further decomposed into pieces P and Q. We bound the space used by all Voronoi diagrams created for R. Recall

that every Voronoi diagram and point location structure corresponds to a node u of P and a hole of Q, or vice versa. The size of each additively weighted Voronoi diagram stored for a node of P is $O(b(Q))$, so $O(n(P)b(Q))$ for all nodes of P. The total space for each piece R is thus $O(n(R)b(R))$ plus $O(n(P)b(Q) + n(Q)b(P))$ if R is decomposed into P and Q. Using Lemma 1, it can be shown that the sum of $O(n(R)b(R))$ over all pieces R is bounded by $O(n^{3/2})$ (The same reasoning can be applied to bound $O(n(P)b(Q) + n(Q)b(P))$).

To compute the distance from u to v, we traverse the recursive decomposition starting from the piece that corresponds to the whole initial graph G. Suppose that the current piece is $R = (V, E)$, which is partitioned into P and Q with a Jordan curve separator C. If $v \in C$ then, because the nodes of C are boundary nodes in both P and Q, we return the additive weight $\omega(v)$ in the Voronoi diagram stored for u, which is equal to the distance from u to v in G. Similarly, if $u \in C$ then we retrieve and return the distance from u to v in the whole G. The remaining case is that both u and v belong to a unique piece P or Q. If both u and v belong to the same piece on the lower level of the decomposition, we continue to that piece. Otherwise, assume without loss of generality that $u \in Q$ and $v \in P$. Then, the shortest path from u to v must go through a boundary node v_i of P. We therefore perform a point location query for v in each of the Voronoi diagram stored for u and for some hole h of P. Let $s_1 \dots s_g$ be the sites returned by these queries, where $g = \mathcal{O}(1)$ is the number of holes of P. The distance in G from u to s_i is $\omega(s_i)$, and the distance in P from s_i to v is stored in $T^P_{s_i}$. We compute the sum of these two terms for each s_i, and return the minimum sum computed.

The query time is $\mathcal{O}(\log n)$ since at each step of the traversal, we either descend to a smaller piece in $\mathcal{O}(1)$ time or terminate after having found the desired distance in $\mathcal{O}(\log n)$ time by $\mathcal{O}(1)$ queries to the point location structure.

Remark. Part of Theorem 9 is proved in [25], though not stated there explicitly as a theorem. It is a tradeoff to Theorem 1.1 of [25], requiring less space, and hence more applicable to our problem.

3 Near linear space data structure for any number of failures

In this section we show how to adapt the approach of [18] for dynamic distance oracles supporting cumulative edge changes to support distance queries with failed vertices. The main technical challenge is in dealing with failures of high-degree vertices, since such vertices may belong to many pieces at each level of the decomposition. For example, think of a failure of the central vertex in a wheel graph, which belongs to all the pieces in the recursive decomposition. Note that standard degree reduction techniques such as vertex splitting are not useful because when a vertex fails all its copies fail. This is in contrast with the situation when dealing only with edge-weight updates, since each edge can be in at most one piece per level. We circumvent this by defining and employing the *strictly internal dense distance graph*. The main intuition is that strictly internal DDGs enable us to handle pieces that only contain failed boundary vertices, i.e. do not contain any internal vertex that fails. Then, only pieces that contain internal failed vertices are "problematic". Note however, that a vertex is internal in at most one piece per level of the decomposition.

Definition 10. *The* strictly internal *dense distance graph of a piece P, denoted DDG°_P, is a complete directed graph on the boundary vertices of P. An edge (u, v) has weight $d^{\circ}_P(u, v)$ equal to the length of the shortest u-to-v path in P that is internally disjoint from ∂P.*

The sole difference to the standard definition is that in our case paths are not allowed to go through ∂P. Observe that the shortest path in P between two vertices of ∂P is still represented in DDG°_P, just not necessarily by a single edge as in DDG_P. This establishes the following lemma.

Lemma 11. *For any piece P and any two boundary vertices $u, v \in \partial P$, the u-to-v distance in DDG°_P equals the u-to-v distance in DDG_P.*

We now discuss how to efficiently compute DDG°_P. We construct a planar graph $\hat{P}$,

by creating a copy of P and incrementing the weight of each edge uv, such that $u \in \partial P$, by $C = 2\sum_{e \in E(G)} |w(e)|$. $DDG_{\hat{P}}$ can be computed in $\mathcal{O}(|\partial P|^2 + |P| \log |P|)$ time using MSSP [31, 7]. Observe that any u-to-v path in $\hat{P}$ that is internally disjoint from $\partial \hat{P}$ has exactly one edge uw with $u \in \partial P$, so its length is at least C and less than $2C$, while any u-to-v path that has an internal vertex in ∂P is of length at least $2C$. Therefore, the u-to-v distance in $\hat{P}$ is equal to C plus the length of the shortest u-to-v path in P that is internally disjoint from ∂P if the latter one is not ∞. We thus set $d^{\circ}_P(u, v) = d_{\hat{P}}(u, v) - C$. This completes the description of the computation of DDG°_P. Note that since C is defined in terms of G rather than P, edge weights greater than C in DDG°_P effectively represent infinite length in the sense that such edges will never be used by any shortest path (in P nor in G). Also note that it follows directly from the definition of the Monge property that subtracting C from each entry of a Monge matrix preserves the Monge property. Therefore, we can use $\bigcup_P DDG^{\circ}_P$ in FR-Dijkstra (Theorem 4) instead of $\bigcup_P DDG_P$.

Preprocessing. We compute a complete recursive decomposition tree $\mathcal{T}$ of G in time $\mathcal{O}(n)$ as discussed in Section 2. We compute DDG°_P for each non-leaf piece $P \in \mathcal{T}$ and preprocess it as in FR-Dijkstra. By Proposition 2, Theorem 4 and the above discussion, the time and space complexities are $\mathcal{O}(n \log^2 n)$ and $\mathcal{O}(n \log n)$ respectively.

Query. Upon query (u, v, X), for each $i \in \{u, v\} \cup X$ we arbitrarily choose a leaf-piece P_i containing i, and run FR-Dijkstra on the union of the following DDG°s, which we denote by $\mathcal{D}$ (inspect Fig. 2 for an illustration):

1. For each $w \in \{u, v\}$, $DDG^{\circ}_{P_w}$ of $P_w \setminus X$ with w regarded as a boundary vertex. This can be computed on the fly in constant time since the size of the leaf piece P_w is constant.

2. For each $w \in \{u, v\}$, for each ancestor P of P_w (including P_w), DDG°_Q of the sibling Q of P if Q does not contain an internal (i.e. non-boundary) vertex of X.

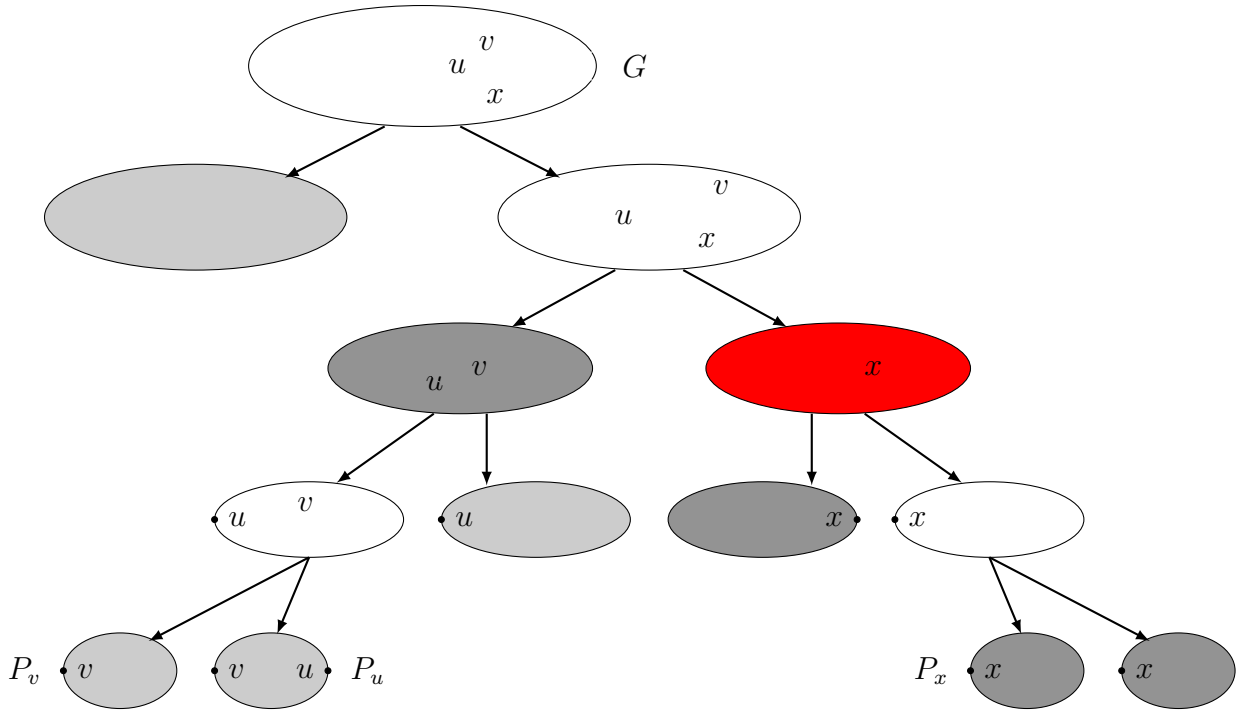


Figure 2: A complete recursive decomposition tree $\mathcal{T}$ of a graph G. The light gray and red pieces are the ones that would be considered by the failure-free distance oracle. However, given the failure of vertex x, the DDG° of the red piece is invalid. The dark gray pieces are the ones that our algorithm considers instead. They allow us to represent the DDG° of the red piece subject to the failure of x.

3. For each $x \in X$, $DDG^\circ_{P_x}$ of $P_x \setminus X$. This can be computed on the fly in constant time since the size of the leaf piece P_x is constant.

4. For each $x \in X$, for each ancestor P of P_x (including P_x), DDG°_Q of the sibling Q of P if Q does not contain an internal vertex of X.

We can identify these DDG°s in $\mathcal{O}(k \log n)$ time by traversing the parent pointers from each P_i, for $i \in X$, and marking all the nodes that have an internal failed vertex. We make one small but crucial change to FR-Dijkstra. When running FR-Dijkstra, we do not relax edges whose tail is a failed vertex. This guarantees that, although failed vertices might appear in the graph on which FR-Dijkstra is invoked, the u-to-v shortest path computed by FR-Dijkstra does not contain any failed vertices. We therefore obtain the following lemma.

Lemma 12. *There exists a data structure of size $\mathcal{O}(n \log n)$, which can be constructed in $\mathcal{O}(n \log^2 n)$ time, and answer the following queries in $\mathcal{O}(\sqrt{kn} \log^2 n)$ time. Given vertices u and v, and a set X of k failing vertices, report the length of a shortest u-to-v path in that avoids the vertices of X.*

Proof. We have already discussed the space occupied by the oracle and the time required to build it. It remains to analyze the query algorithm.

Correctness. First, it is easy to see that no edge (y, z) of any of the DDG°s in $\mathcal{D}$ represents a path containing a vertex $x \in X$, unless $\{y, z\} \cap X \neq \emptyset$. The latter case does not affect the correctness of the algorithm, since in FR-Dijkstra we do not relax edges whose tail is a failed vertex. Hence, the algorithm never computes a distance corresponding to a path going through a failed vertex.

It remains to show that the shortest path in $G \setminus X$ is represented in $\mathcal{D}$. For this, by Corollary 7, it suffices to prove that for each piece A in the cone of u (and similarly in the cone of v), either DDG°_A for $A \setminus X$ belongs to $\mathcal{D}$, or $\mathcal{D}$ contains enough information to reconstruct DDG°_A for $A \setminus X$ (i.e. subject to the failures) during FR-Dijkstra. In the latter case we say that DDG°_A is *represented* in $\mathcal{D}$. Note that, for any piece P, DDG°_P is represented in $\mathcal{D}$ if the DDG°s of its two children in $\mathcal{T}$ are represented in $\mathcal{D}$. (This follows by an argument identical to the one used in the proof of Lemma 6.) If A contains no internal failed vertex then DDG°_A is in $\mathcal{D}$ by point 1 or 2 above. We next consider the case that A does contain some failed vertex $x \in X$ as an internal vertex. Thus A is an ancestor of P_x. To show that A is represented in $\mathcal{D}$, we prove that for any failed vertex $y \in X$, the DDG° of any non-root ancestor of P_y in $\mathcal{T}$ is represented in $\mathcal{D}$.

We proceed by the minimal counterexample method. For any $y \in X$, $DDG^\circ_{P_y}$ is in $\mathcal{D}$ since it is computed on the fly in point 3. Let F be the deepest node in $\mathcal{T}$ that is a strict ancestor of P_y for some $y \in X$ and whose DDG° is not represented in $\mathcal{D}$. It follows that one of F's children must also be an ancestor of P_y and by the choice of F its DDG° is represented in $\mathcal{D}$. Let the other child of F be J. If J is an ancestor of some P_z, $z \in X$, then DDG°_J is

also represented in $\mathcal{D}$ by the choice of F. Otherwise, J does not contain any internal failed vertex, and hence DDG°_J is in $\mathcal{D}$ by point 4. In either case, the DDG°s of both children of F are represented in $\mathcal{D}$, so DDG°_F is also represented in $\mathcal{D}$, a contradiction.

Time complexity. Let $r = n/k$ and consider an r-division of G in $\mathcal{T}$. The pieces of this r-division have $\mathcal{O}(\frac{n}{\sqrt{r}}) = \mathcal{O}(\sqrt{kn})$ boundary vertices in total and this is known to also be an upper bound on the total number of boundary vertices (with multiplicities) of ancestors of pieces in this r-division (cf. the discussion after Corollary 5.1 in [25]).

Recall that we have chosen a leaf-piece P_i for each vertex $i \in \{u, v\} \cup X$. Each piece (other than the P_is) whose DDG° belong to $\mathcal{D}$ is a siblings of an ancestor of some P_i. This implies that each $i \in \{u, v\} \cup X$ contributes the DDG°s of at most two pieces per level of the decomposition. Let the ancestor of P_i that is in the r-division be R_i. For each P_i, we only need to bound the total size of pieces it contributes that are descendants of R_i, since we have already bounded the total size of the rest. We do so by applying Lemma 1 for the subtree of $\mathcal{T}$ rooted at each R_i. (The extra $\mathcal{O}(\sqrt{r})$ boundary vertices we start with do not alter the analysis of this lemma as these many are anyway introduced by the first separation of R_i.) It yields $2\sum_\ell \frac{\sqrt{r}}{c_2^\ell}$, where $c_2 > 1$, which is $\mathcal{O}(\sqrt{r})$. Summing over all $k+2$ pieces P_i we obtain the upper bound $\mathcal{O}(k\sqrt{r}) = \mathcal{O}(\sqrt{kn})$.

FR-Dijkstra runs in time proportional to the total number of vertices of the DDG°s in $\mathcal{D}$ up to a $\log^2 n$ multiplicative factor and hence the time complexity follows. □ □

Remark. By using existing techniques (cf. [30, Section 5.4]), we can report the actual shortest path ρ in time $\mathcal{O}(|\rho| \log\log \Delta_\rho)$, where Δ_ρ is the maximum degree of a vertex of ρ in G.[4]

[4]This remark also applies to the dynamic distance oracle presented in Section 6. However, it does not apply to the oracles presented in Section 4, where we use a different modification of DDGs for which we can not afford to store the MSSP data structures that would allow us to return the actual shortest paths efficiently.

4 Tradeoffs

In this section we describe a tradeoff between the size of the oracle and the query-time. We first define another useful modification of dense distance graphs.

Definition 13. *The strictly external dense distance graph of G for pieces $P_1, \ldots, P_i$ ($DDG^{\circ}_{ext}(P_1, \ldots, P_i)$) is a complete directed graph on the boundary vertices of $P_1, \ldots, P_i$. The edge (u, v) has weight equal to the length of the shortest u-to-v path in $G \setminus \big(\big(\bigcup_{j=1}^{i} P_j\big) \setminus \{u, v\}\big)$.*

DDG°_{ext}s can be preprocessed using Theorem 4 together with DDG°s so that we can perform efficient Dijkstra computations in any union of DDG°_{ext}s and DDG°s.

The number of pieces in an r-division is at most cn/r for some constant c. For convenience, we define

$$g(n, r, k) = \binom{cn/r}{k} \leq \frac{(cn)^k}{r^k k!} \leq \frac{n^k}{r^k k},$$

where the last inequality holds for sufficiently large k and we use it throughout, to hide factors that are solely dependent on k.

4.1 The case of a single failure

For ease of presentation we first describe an oracle that can handle just a single failure. We prove the following lemma, which is a restricted version of our main result, Theorem 15.

Lemma 14. *For any $r \in [1, n]$, there exists a data structure of size $\mathcal{O}(\frac{n^{5/2}}{r^{3/2}} + n \log^2 n)$, which can be constructed in time $\tilde{\mathcal{O}}(\frac{n^{5/2}}{r^{3/2}} + n^2)$, and can answer the following queries in $\mathcal{O}(\sqrt{r} \log^2 n)$ time. Given vertices u, v, x, report the length of a shortest u-to-v path that avoids x.*

We first perform the precomputations of Section 3. We also obtain an r-division of G from $\mathcal{T}$ in $\mathcal{O}(n)$ time. Let us denote the pieces of this r-division by $R_1, \ldots, R_q$.

Warm up. We first show how to get an $\mathcal{O}(\frac{n^3}{r^2})$-space oracle with $\tilde{\mathcal{O}}(\sqrt{r})$ query time for a single failure using the approach of Section 3. For each triplet R_i, R_j, R_k of pieces in the

r-division we store $DDG^{\circ}_{ext}(R_i, R_j, R_k)$; these require space $\mathcal{O}(g(n,r,3)(\sqrt{r})^2) = \mathcal{O}(\frac{n^3}{r^2})$ in total. Given u, v, x in R_u, R_v and R_x, respectively, we consider the required DDG°s that allow us to represent $DDG^{\circ}_{R_j}$ subject to the failures for each j as in Section 3 (i.e. the DDG°s in items 2 and 4 in Section 3 are only taken for ancestors of P_i that are descendants of R_j). We then run FR-Dijkstra on these along with $DDG^{\circ}_{ext}(R_u, R_v, R_x)$, not relaxing edges whose tail is x if encountered. This takes time $\mathcal{O}(\sqrt{r}\log^2 n)$.

Main Idea for reducing the space complexity. Instead of storing information for triplets of pieces, we will store more information, but just for pairs. Given u, v, x we show how to compute $d(u, v, x)$ relying on the information stored for the pair of pieces R_u and R_x. We first compute the distances from u to each $w \in \partial R_u \cup \partial R_x$ in $G \setminus \{x\}$ using FR-Dijkstra with $DDG^{\circ}_{ext}(R_u, R_x)$ as in the warm up above. We then identify an appropriate piece Q in $\mathcal{T}$ that contains v, and does not contain u nor x. Exploiting the fact that distances within Q remain unchanged when x fails, we employ Voronoi Diagrams with point location for the piece Q, adapting ideas from [25].

Additional Preprocessing. For each pair of pieces (R_i, R_j) of the r-division we compute and store the following:

1. $DDG^{\circ}_{ext}(R_i, R_j)$.
2. Let S be a separator in the recursive decomposition, separating a piece into two subpieces Q and R, such that $R_i \subseteq R$ and $R_j \not\subset Q$. For each $y \in \partial R_i \cup \partial R_j$, for each hole h of Q, we compute and store a Voronoi diagram with the point location data structure for Q, with sites the boundary vertices of Q that lie on h, and additive weights the distances from y to these sites in $G \setminus ((R_i \cup R_j) \setminus \{y\})$.

We now show that the space required is $\mathcal{O}(\frac{n^{5/2}}{r^{3/2}} + n\log^2 n)$. The space required for the preprocessed internal and external dense distance graphs is $\mathcal{O}(n \log n)$ and $\mathcal{O}(\frac{n^2}{r})$, respectively, by Theorem 4. We next analyze the space required for storing the Voronoi diagrams. We

consider $\mathcal{O}(g(n,r,2)) = \mathcal{O}(\frac{n^2}{r^2})$ pairs of pieces (R_i, R_j), and for each of the $\mathcal{O}(\sqrt{r})$ boundary vertices of each such pair we store, in the worst case, a Voronoi diagram for each of the $\mathcal{O}(1)$ holes of each sibling of the nodes in the root-to-R_i and root-to-R_j paths in $\mathcal{T}$. The total number of sites of all Voronoi diagrams we store for a pair of pieces can be upper bounded by $\mathcal{O}(\sqrt{n})$ by noting that the number of sites at level ℓ of T_G has $\mathcal{O}(\sqrt{n}/c_2^\ell)$ boundary vertices by Lemma 1. By Theorem 9, the space required to store a representation of a set of Voronoi diagrams with the functionality allowing for efficient point location queries for a piece P, with sites a subset of the boundary vertices of P, lying on a hole h is $\mathcal{O}\big(\sum_{P\in\mathcal{T}}(\mathcal{S}_{P,h} + |P|\log|P|)\big)$, where $\mathcal{S}_{P,h}$ is the total cardinality of these sets of sites. Summing over all holes of all pieces P, noting that $\sum_{P\in\mathcal{T}}\sum_h \mathcal{S}_{P,h} = \mathcal{O}(\frac{n^{5/2}}{r^{3/2}})$ by the above discussion, and using Proposition 2, the total space required for all Voronoi diagrams is $\mathcal{O}(\frac{n^{5/2}}{r^{3/2}} + n\log^2 n)$.

We analyze the construction time in Section 5. The internal dense distance graphs can be computed in time $\mathcal{O}(n\log^2 n)$. The external dense distance graphs and the additive weights can be computed in time $\mathcal{O}(\frac{n^2}{r}\log^2 n)$ and $\mathcal{O}(\frac{n^2}{r}\sqrt{nr}\log^3 n)$, respectively; see Lemmas 16 and 17. We show in Lemma 18 that we can compute all required Voronoi diagrams in time $\tilde{\mathcal{O}}(n^2 + \mathcal{S})$, where $\mathcal{S}$ is the size of their representation described in Section 2.

Query. If any two of $\{u, v, x\}$ are in the same piece of the r-division, then we can use FR-Dijkstra taking into account just two pieces of the r-division containing u, v, and x, similarly to the description in the warm up above. We therefore assume no two of $\{u, v, x\}$ are in the same piece of the r-division. We first retrieve a piece R_v of the r-division, containing v (to support that, each vertex stores a pointer to some piece of the r-division that contains it). In the following we will need to check whether a vertex is in some particular piece of $\mathcal{T}$. This can be done in $\mathcal{O}(\log n)$ time by storing, for each piece in $\mathcal{T}$, a binary tree with the vertices in the piece. We then proceed as follows (inspect Fig. 3 for an illustration).

1. Following parent pointers of R_v in $\mathcal{T}$, we find the highest ancestor Q of R_v containing neither u nor x. Thus, the sibling R of Q in $\mathcal{T}$ contains a vertex $i \in \{u, x\}$. We find a

descendant R_i of R that is in the r-division and contains i. We then find any piece R_j of the r-division containing the element of $\{u, x\} \setminus \{i\}$. Note that, by choice of Q, R_j is not a descendant of Q. Finding these pieces requires time $\mathcal{O}(\log^2 n)$.

(a) The DDG°s that allow us to represent $DDG^\circ_{R_u}$ as in Section 3.

(b) $DDG^\circ_{ext}(R_u, R_x)$;

(c) the DDG°s that allow us to represent $DDG^\circ_{R_x}$ subject to the failure of x as in Section 3.

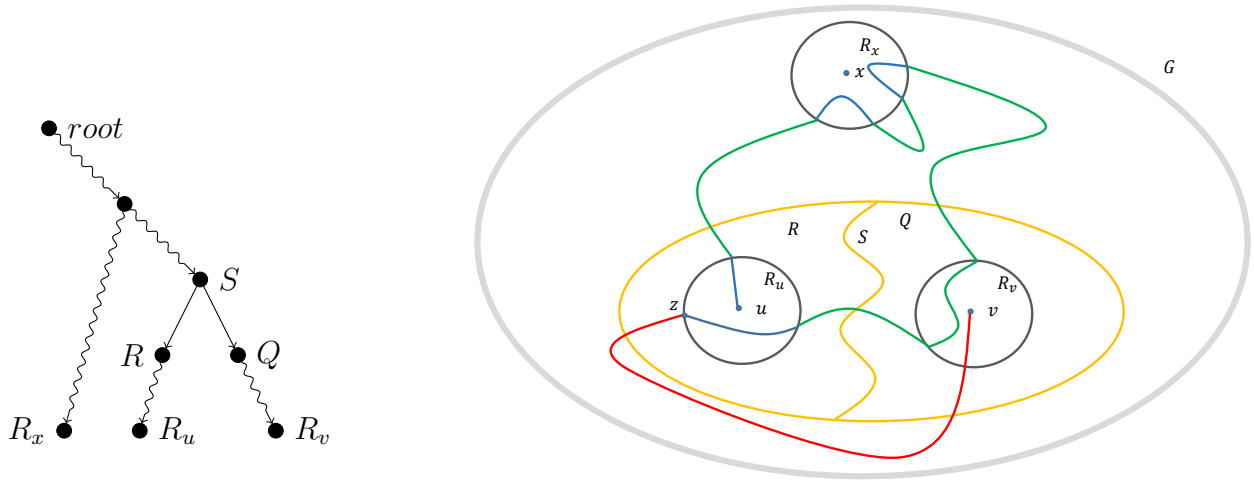
This takes time $\mathcal{O}(\sqrt{r}\log^2 n)$ and returns $d_G(u, y, x)$ for each $y \in \partial R_u \cup \partial R_x$.

2. For each $y \in (\partial R_u \cup \partial R_x) \setminus \{x\}$, for each hole h of Q, we perform an $\mathcal{O}(\log^2 n)$-time query to the Voronoi diagram stored for R_u, R_x, y, and h to get the distance from y to v in $G \setminus ((R_u \cup R_x) \setminus \{y\})$. The required distance is the minimum $d(u, y, x) + d(y, v, (R_u \cup R_x) \setminus \{y\})$ over all y. Each query takes $\mathcal{O}(\log^2 n)$ time and hence the total time required is $\mathcal{O}(\sqrt{r}\log^2 n)$.

We now argue the correctness of the query algorithm. Let ρ be a shortest u-to-v path that avoids x. Let z be the last vertex of ρ that belongs to $\partial R_u \cup \partial R_x$. Let h' be the hole of Q such that the last vertex of ρ that belongs to the boundary of Q belongs to hole h'. The distance $d_G(u, z, x)$ from u to z in $G \setminus \{x\}$ is computed by the FR-Dijkstra computation in step 2, while the distance from z to v in $G \setminus \{x\}$ is obtained from the query to the Voronoi diagram stored for R_u, R_x, z, and h'. It is easy to see that we do not obtain any distance that does not correspond to an actual path in $G \setminus \{x\}$ and hence the correctness of the query algorithm follows.

4.2 Handling multiple failures

The warm-up approach of Section 4.1 can be trivially generalized to handle k failed vertices by considering $(k+2)$-tuples of pieces of the r-division. (We consider the elements of tuples



(a) root-to-R_i paths in $\mathcal{T}$

(b) The u-to-v path in $G \setminus \{x\}$

Figure 3: To the left: A view of the root-to-R_i paths in $\mathcal{T}$. Straight edges denote edges of the tree, while snake-shaped edges denote paths. To the right: A view of the shortest path in G. The paths in blue are represented by the DDG°s, the ones in green by DDG°_{ext} and the length of the one in red is returned by the point location query in the Voronoi diagram.

to be unordered throughout.) The space required is $\tilde{\mathcal{O}}(g(n, r, k+2)(\sqrt{r})^2) = \tilde{\mathcal{O}}(\frac{n^{k+2}}{r^{k+1}})$ and queries can be answered in $\tilde{\mathcal{O}}(k\sqrt{r})$ time. We reduce the space to $\tilde{\mathcal{O}}(\frac{n^{k+1}}{r^{k+1}}\sqrt{nr})$ by generalizing the main algorithm of Section 4.1.

Preprocessing.

1. We perform the precomputations of Section 3.

2. For each $(k+1)$-tuple of pieces $(R_{i_1}, \ldots, R_{i_{k+1}})$ of the r-division we compute and store the following:

   (a) $DDG^{\circ}_{ext}(R_{i_1}, \ldots, R_{i_{k+1}})$.

   (b) Let S be a separator in the recursive decomposition, separating a piece into Q and R, such that for some j $R_{i_j} \subseteq R$ and none of the other pieces of the tuple is a subgraph of Q. For each $y \in \bigcup_{j=1}^{k+1} \partial R_{i_j}$, for each hole h of Q, we store a Voronoi diagram with the point location data structure for Q, with sites the boundary

vertices of Q that lie on h, and additive weights the distances from y to these sites in $G \setminus \big(\big(\bigcup_{j=1}^{k+1} R_{i_j}\big) \setminus \{y\}\big)$.

Query. We first retrieve a piece R_v of the r-division, containing v. We can again assume that no two elements of $\{u\} \cup X$ are in the same piece of the r-division, since otherwise we can answer the query in $\mathcal{O}(k\sqrt{r})$ time by running FR-Dijkstra on the DDG°_{ext} of a $(k+1)$-tuple and the DDG°s we add for each of the pieces in the tuple, following the algorithm of Section 3.

The algorithm is then essentially the same as that of Section 4.1.

1. We find the highest ancestor Q of R_v in $\mathcal{T}$ that does not contain any of the elements of $\{u\} \cup X$ and retrieve a descendant of its sibling in the r-division that does contain some element $i \in \{u\} \cup X$. We then identify a piece R_j in the r-division for each $j \in \{u\} \cup X \setminus \{i\}$. This requires time $\mathcal{O}(k \log^2 n)$.

2. We run FR-Dijkstra on DDG°s of total size $\mathcal{O}(k\sqrt{r})$.

3. We perform $\mathcal{O}(k\sqrt{r})$ point location queries to Voronoi diagrams of Q, each requiring time $\mathcal{O}(\log^2 n)$.

We hence obtain the general tradeoff theorem.

Theorem 15. *For any integer $r \in [1, n]$ and for any integer $k \leq \frac{n}{r}$, there exists a data structure of size $\mathcal{O}(\frac{(cn)^{k+1}}{r^{k+1}} \frac{1}{k!} \sqrt{nkr} + n \log^2 n)$, which can be constructed in time $\tilde{\mathcal{O}}(\frac{(cn)^{k+1}}{r^{k+1}} \frac{1}{k!} \sqrt{nkr} + n^2)$, for some constant $c > 1$, and can answer the following queries in $\mathcal{O}(k\sqrt{r} \log^2 n)$ time. Given vertices u and v and a set X of at most k failing vertices, report the length of a shortest u-to-v path that avoids X.*

Remark. Our distance oracle can handle any number f of failures that lie in at most k pieces of the r-division in time $\tilde{\mathcal{O}}((k+\sqrt{fk})\sqrt{r})$ with an $\tilde{\mathcal{O}}(\frac{n^{k+1}}{r^{k+1}}\sqrt{nr})$-size oracle. This follows from the fact that the DDG°s we will add for a piece with f_i failures have total size $\tilde{\mathcal{O}}(\sqrt{f_i r})$ by the same analysis as in the proof of Lemma 12 and the fact that, given $f_1, \ldots, f_k$ such that $\sum_{i=1}^{k} f_i = f$, we have $\sum_{i=1}^{k} \sqrt{f_i} \leq \sqrt{fk}$ by the Cauchy-Schwarz inequality.

Proof of Theorem 15. The correctness of the query algorithm follows by an argument identical to the one for the case of single failures (see Section 4.1); its time complexity is analyzed above. We next analyze the space required by our data structure and its construction time.

Space Complexity. The space occupied by the preprocessed DDG°s and DDG°_{ext}s is $\mathcal{O}(n \log n)$ and $\mathcal{O}(g(n,r,k+1)k^2 r) = \mathcal{O}(\frac{(cn)^{k+1}}{r^{k+1}} \frac{kr}{k!})$, respectively, by Theorem 4.

We bound the space required for the Voronoi diagrams by $\mathcal{O}(g(n,r,k+1)k\sqrt{nkr} + n\log^2 n)$ as follows. For each of the of the $\mathcal{O}(k\sqrt{r})$ boundary vertices of each of the $\mathcal{O}(g(n,r,k+1))$ $(k+1)$-tuples, we store a Voronoi diagram for each of the $\mathcal{O}(1)$ holes, of (at most) each of the siblings of the nodes in the root-to-R_i path in $\mathcal{T}$ for each R_i in the tuple. With an argument identical to the one used in the proof of Theorem 4, the total number of boundary vertices (with multiplicities) of all of these pieces is $\mathcal{O}(\sqrt{kn})$. Hence the total number of all Voronoi diagrams that we store is $\mathcal{O}(g(n,r,k+1)k\sqrt{nkr})$. By Theorem 9, the size required to store them, with the required functionality, is $\mathcal{O}(g(n,r,k+1)k\sqrt{nkr} + \sum_{P\in\mathcal{T}} |P| \log |P|) = \mathcal{O}(\frac{(cn)^{k+1}}{r^{k+1}} \frac{1}{k!}\sqrt{nkr} + n\log^2 n)$, where the last equality follows by Proposition 2.

The total space is thus $\mathcal{O}(\frac{(cn)^{k+1}}{r^{k+1}} \frac{1}{k!})(kr + \sqrt{nkr}) + n\log^2 n) = \mathcal{O}(\frac{(cn)^{k+1}}{r^{k+1}} \frac{1}{k!}\sqrt{nkr} + n\log^2 n)$ since $k \leq n/r$.

Preprocessing time. We compute the DDG°_{ext}s and the required additive weights of all $(k+1)$-tuples in time $\tilde{\mathcal{O}}(\frac{(cn)^{k+1}}{r^k} \frac{1}{(k-1)!})$ and $\tilde{\mathcal{O}}(\frac{(cn)^{k+1}}{r^{k+1}} \frac{1}{(k-1)!}\sqrt{nkr})$, respectively, using Lemmas 16 and 17. Finally, constructing the Voronoi diagrams requires time $\tilde{\mathcal{O}}(n^2 + \mathcal{S})$, where $\mathcal{S}$ is the total size of their representation, which is equal to the total number of sites in these diagrams (with multiplicities), as shown in Lemma 18; this dominates the time complexity. □

5 Efficient preprocessing

In this section we show how to efficiently compute the data structures described in Section 4. It is shown in [32] (cf. see Theorem 3 therein) that, given a geometrically increasing sequence

of numbers $\nabla = (r_1, r_2, \ldots, r_\nu)$, where r_1 is a sufficiently large constant, $r_{i+1}/r_i = b$, for all i, for some constant $b > 1$, and $r_\nu = n$, we can obtain r-divisions for all $r \in \nabla$ in time $\mathcal{O}(n)$ in total. These r-divisions satisfy the property that a piece in the r_i-division is a —not necessarily strict— descendant (in $\mathcal{T}$) of a piece in the r_j-division for each $j > i$.

We first show how to efficiently compute the external DDGs for all k-tuples of pieces of an r-division, $r \in \nabla$.

Lemma 16. *Given $r_i \in \nabla$ and an integer $d \leq \frac{n}{r_i}$, one can compute DDG°_{ext} for all d-tuples of pieces of each r_t-division, $t \geq i$, in time $\mathcal{O}(\frac{(cn)^d}{r_i^{d-1}} \frac{1}{(d-2)!} \log^2 n)$ for some constant $c > 1$.*

Proof. We prove this lemma by induction on ∇ from top to bottom. For $r_\nu = n$, the only piece is G. $DDG^\circ_{ext}(G)$ is the empty graph. Assume inductively that we have $DDG^\circ_{ext}(R_1, \ldots, R_d)$ for every d-tuple $(R_1, \ldots, R_d)$ of pieces at the r_{i+1}-division. Let $Q_1, \ldots, Q_d$ be pieces at the r_i-division. Note that every piece at level r_i is contained in some piece at level r_{i+1}, but a piece at level r_{i+1} might contain multiple pieces at level r_i. Let $R_1, \ldots, R_d$ be pieces of the r_{i+1}-division such that each Q_j is a subgraph of some $R_{j'}$. Let $\mathcal{Q}_{R_j}$ be the maximal subset of $\{Q_1, \ldots, Q_d\}$ such that each piece in $\mathcal{Q}_{R_j}$ is contained in R_j. For every $j \in \{1, \ldots d\}$ let $R'_j = R_j \setminus (\bigcup \mathcal{Q}_{R_j})$ (i.e. the allowed internal part of R_j). Since R_j and each $Q_m \in \mathcal{Q}_{R_j}$ have $\mathcal{O}(\sqrt{r_{i+1}})$ and $\mathcal{O}(\sqrt{r_i})$ boundary vertices respectively, R'_j has $\mathcal{O}(\sqrt{r_{i+1}} + \sqrt{r_i}|\mathcal{Q}_{R_j}|) = \mathcal{O}(|\mathcal{Q}_{R_j}|\sqrt{r_{i+1}})$ boundary vertices (recall that $r_{i+1}/r_i = b$).

We compute $DDG^\circ_{R'_j}$ in a similar manner to the query of Section 3 by running FR-Dijkstra on the union of the following DDG°s. For each piece $Q_m \in \mathcal{Q}_{R_j}$, for each ancestor Q of Q_m (including Q_m) that is a strict descendant of R_j in $\mathcal{T}$, we take the DDG°_P of the sibling P of Q if P contains no piece of $\mathcal{Q}_{R_j}$. The pieces of $\mathcal{Q}_{R_j}$ have $\mathcal{O}(|\mathcal{Q}_{R_j}|\sqrt{r_i})$ boundary vertices in total and the total number of boundary vertices for their ancestors is bounded by $\mathcal{O}(|\mathcal{Q}_{R_j}|\sqrt{r_{i+1}})$. Running FR-Dijkstra from each of the $\mathcal{O}(|\mathcal{Q}_{R_j}|\sqrt{r_{i+1}})$ boundary vertices of R'_j yields $DDG^\circ_{R'_j}$ and requires $\mathcal{O}(|\mathcal{Q}_{R_j}|\sqrt{r_{i+1}}|\mathcal{Q}_{R_j}|\sqrt{r_{i+1}} \log^2 n) = \mathcal{O}(|\mathcal{Q}_{R_j}|^2 r_{i+1} \log^2 n)$ time in total. When summing over $R_1, \ldots, R_d$ we get $\sum_{j=1}^d |\mathcal{Q}_{R_j}|^2 r_{i+1} \log^2 n \leq r_{i+1} \log^2 n \big(\sum_{j=1}^d |\mathcal{Q}_{R_j}|\big)^2 = d^2 r_{i+1} \log^2 n$. The inequality is due to the Cauchy-Schwarz inequality and the equality follows

from the fact that $\sum_{j=1}^{d} |\mathcal{Q}_{R_j}| = d$.

Let $\mathcal{D} = DDG^{\circ}_{ext}(R_1, \ldots, R_d) \bigcup (\bigcup_{j=1}^{d} DDG^{\circ}_{R'_j})$. $DDG^{\circ}_{ext}(R_1, \ldots, R_d)$ and $\bigcup_{j=1}^{d} DDG^{\circ}_{R'_j}$ contribute $\mathcal{O}(d\sqrt{r_{i+1}})$ and $\mathcal{O}(d(\sqrt{r_{i+1}} + \sqrt{r_i}))$ boundary vertices to $\mathcal{D}$, respectively. We run FR-Dijkstra on $\mathcal{D}$ from each boundary vertex of Q_m for $m \in \{1, \ldots d\}$. There are $\mathcal{O}(d\sqrt{r_i})$ such boundary vertices, so this requires $\mathcal{O}(d\sqrt{r_i}d(\sqrt{r_{i+1}} + \sqrt{r_i}) \log^2 n) = \mathcal{O}(d^2 r_{i+1} \log^2 n)$ time, and yields $DDG^{\circ}_{ext}(Q_1, \ldots, Q_d)$.

We can thus compute $DDG^{\circ}_{ext}(Q_1, \ldots, Q_d)$ for all d-tuples at level r_i in $\mathcal{O}((g(n, r_i, d)d^2 r_{i+1} \log^2 n) = \mathcal{O}(\frac{(cn)^d}{r_i^d} r_{i+1} \frac{1}{d!} d^2 \log^2 n) = \mathcal{O}(\frac{(cn)^d}{r_i^{d-1}} \frac{1}{(d-2)!} \log^2 n)$ time, assuming that we have the DDG°_{ext}s for all d-tuples of pieces of r_t-divisions, $t > i$.

The time to compute the DDG°_{ext}s for all d-tuples of pieces of all r_t-divisions, $t > i$, is, inductively, $\mathcal{O}\big((cn)^d \frac{1}{(d-2)!} \log^2 n \sum_{t=i+1}^{\nu} \frac{1}{r_t^{d-1}}\big)$, and $\sum_{t=i+1}^{\nu} \frac{1}{r_t^{d-1}} = \frac{1}{r_i^{d-1}} \sum_{t=1}^{\nu-i} (\frac{1}{b^{d-1}})^t = \mathcal{O}(\frac{1}{r_i^{d-1}})$ since $b^{d-1} > 1$. Thus computing the DDG°_{ext}s for d-tuples of pieces of the r_i-division dominates the time complexity. □

We next show how to efficiently compute the additive distances with respect to which the Voronoi diagrams stored by our oracle are computed.

Lemma 17. *Let $\mathcal{R}_r$ be an r-division, such that $r \in \nabla$, and let $d \leq \frac{n}{r}$ be an integer. For all d-tuples of pieces $R_1, \ldots, R_d$ in $\mathcal{R}_r$ and for all pieces $Q \in \mathcal{T}$ such that Q does not contain any of the pieces R_i, and Q is a sibling of a node in the root to-R_i path in $\mathcal{T}$ for some R_i, one can compute the distances from each $y \in \bigcup_{i=1}^{d} \partial R_i$ to each boundary vertex of Q in the graph $G \setminus \big(\big(\bigcup_{i=1}^{d} R_i\big) \setminus \{y\}\big)$ in time $\mathcal{O}\big(\frac{(cn)^d}{r^d} \frac{1}{(d-2)!} \sqrt{ndr} \log^3 n\big)$ in total, for some constant $c > 1$.*

Proof. Let us consider a d-tuple of pieces $(R_1, \ldots, R_d)$ and a piece Q, satisfying the properties in the statement of the lemma. To compute the desired distances, we run FR-Dijkstra from each $y \in \bigcup_{i=1}^{d} \partial R_i$ on the union of the following DDGs:

1. DDG°_{Q}.

2. For each piece $R_i \in \{R_1, \ldots, R_d\}$ for each ancestor A of R_i (including R_i) in $\mathcal{T}$, we take the DDG°_{B} of the sibling B of A if B contains no piece of $R_1, \ldots, R_d$.

This correctly computes the distances by the same arguments that were applied in Section 3. It remains to analyze the time complexity. Consider the (n/d)-division of G in $\mathcal{T}$. By the same argument that was applied in the proof of Lemma 12 we can bound the number of boundary vertices for all the included DDG°s by $\mathcal{O}(\sqrt{dn})$. There are $\mathcal{O}(d\sqrt{r})$ choices of $y \in \bigcup_{i=1}^{d} \partial R_i$, so the time required to run FR-Dijkstra from each y is $\mathcal{O}(d\sqrt{r}\sqrt{dn}\log^2 n) = \mathcal{O}(d\sqrt{nrd}\log^2 n)$.

Each piece $R_i \in \{R_1, \ldots, R_d\}$ has $\mathcal{O}(\log n)$ nodes in the root-to-R_i path in $\mathcal{T}$, hence computing the distances for all possible choices of Q requires time $\mathcal{O}(d^2\sqrt{nrd}\log^3 n)$. Finally, in order to compute the distances for all d-tuples of pieces we need time $\mathcal{O}((g(n,r,d)d^2\sqrt{nrd}\log^3 n) = \mathcal{O}((\frac{(cn)^d}{r^d})\frac{1}{d!}d^2\sqrt{nrd}\log^3 n) = \mathcal{O}((\frac{(cn)^d}{r^d})\frac{1}{(d-2)!}\sqrt{nrd}\log^3 n)$. □

Lemma 18. *We can compute the representation of the Voronoi diagrams described in Section 2 with respect to sets of sites, of total cardinality $\mathcal{S}$, each corresponding to a piece $P \in \mathcal{T}$, consisting of nodes of ∂P that lie on a single hole of P, and specifying an additive weight for each of these nodes in time $\tilde{\mathcal{O}}(n^2 + \mathcal{S})$ in total.*

Proof. We apply Theorem 9 and construct all the Voronoi diagrams corresponding to each of the $\mathcal{O}(1)$ holes of each piece as a batch. For a hole h of a piece P, the time required is $\tilde{\mathcal{O}}(|P||\partial P|^2 + \sum_h \mathcal{S}_{P,h})$, where $\mathcal{S}_{P,h}$ is the total cardinality of the sets of sites corresponding to nodes of ∂P lying on h. Then we have that

$$\sum_{P \in \mathcal{T}} \left(|P||\partial P|^2 + \sum_h |\mathcal{S}_{P,h}|\right) = \mathcal{O}(n^2 + \mathcal{S}),$$

by Propositions 2 and 3 and hence the stated bound follows.

□

6 Dynamic Distance Oracles can handle Vertex Deletions

In this section we briefly explain how the techniques of Section 3, and specifically our notion of strict dense distance graph DDG° can be used to facilitate vertex deletions in dynamic distance oracles for planar graphs. The dynamic distance oracle of [18] for non-negative edge-weight updates was improved and simplified in [31]. In [31], the algorithm obtains an r-division of G, and then computes and preprocesses the DDGs of the pieces of the r-division in $\mathcal{O}(n \log n)$ time to allow for FR-Dijkstra computations in the union of these DDGs in $\mathcal{O}(\frac{n}{\sqrt{r}} \log^2 n)$. For a given query asking for the distance from some vertex u to some vertex v, the algorithm performs standard Dijkstra computations within the piece containing u (resp. v) to compute the distances from u to the boundary vertices of the piece (resp. from the boundary vertices of the piece to v) and combines this with an FR-Dijkstra on the boundary vertices of the r-division. Given an edge update, only the DDG of the unique piece in the r-division containing the updated edge needs to get updated, and this requires $\mathcal{O}(r \log r)$ time. The balance is at $r = n^{2/3} \log^{2/3} n$, yielding $\mathcal{O}(n^{2/3} \log^{5/3} n)$ time per update and query. This result was extended in [28], where the authors showed how to allow for edge insertions (not violating the planarity of the embedding) and edge deletions and further in [30] where the authors showed how to handle arbitrary (i.e. also negative) edge-weight updates. The time complexity was improved by a $\log^{4/3} \log n$ factor in [24].

We observe that, by using DDG°s instead of the standard DDGs, vertex deletions can also be handled as follows. Each vertex is either a boundary vertex in each piece of the r-division containing it, or an internal vertex in a unique piece. If a deleted vertex is a boundary vertex, we just mark it as such and do not relax edges outgoing from it during (FR-)Dijkstra computations. If a deleted vertex is internal, we recompute the DDG° of the piece containing it, and reprocess it in time $\mathcal{O}(r \log r)$ exactly as in the case of edge-weight updates. The only slightly technical issue we need to take into account is that in Section 3,

edge weights in DDG° are shifted by the large constant C (recall that C is defined as twice the sum of edge weights in the entire graph G). The problem is that C might change after each update operation, and this update affects the weights of all the edges in all DDG°s. This can be easily solved using indirection. Instead of using the explicit value of C in each edge weight, we represent C symbolically, and store the actual value of C explicitly at some placeholder. Updating C can be done in constant time because only the explicit value at the placeholder needs to be updated. Whenever an edge weight is required by the algorithm, it is computed on the fly in constant time using the value of C stored in the placeholder. The data structures underlying FR-Dijkstra do not make use of any, for instance, integer data structures like predecessor data structures —informally, all used data structures are comparison based. Hence, since the value of C is greater than all edge-weights at the time they are built, they are identical to the data structures that would have been built for this piece with any subsequent value of C. Vertex additions do not alter shortest paths and hence can be treated trivially. Note that, as in [28], we can afford to recompute the entire data structure from scratch after every $\mathcal{O}(\sqrt{r})$ operations. This guarantees that the number of vertices and number of boundary vertices in each piece remain $\mathcal{O}(r)$ and $\mathcal{O}(\sqrt{r})$, respectively, throughout. We formalize the above discussion in the following theorem.

Theorem 19. *A planar graph G can be preprocessed in time $\mathcal{O}(n\frac{\log^2 n}{\log\log n})$ so that edge-weight updates, edge insertions not violating the planarity of the embedding, edge deletions, vertex insertions and deletions, and distance queries can be performed in time $\mathcal{O}(n^{2/3}\frac{\log^{5/3} n}{\log^{4/3}\log n})$ each, using $\mathcal{O}(n)$ space.*

7 Final Remarks

Perhaps the most intriguing open question related to our results is whether it is possible to answer distance queries subject to even one failure in time $\tilde{\mathcal{O}}(1)$ with an $o(n^2)$-size oracle. Recall that the best known exact failure-free distance oracle that answers queries in $\tilde{\mathcal{O}}(1)$

occupies $\tilde{\mathcal{O}}(n^{3/2})$ space [25]. Note that if the source vertex is fixed, there is a data structure with $\tilde{\mathcal{O}}(n)$ space that supports a single failure in $\tilde{\mathcal{O}}(1)$ time [4]. Expanding this result to support more than one failure is also a subject of interest. Although not mentioned explicitly, the approximate dynamic distance oracle of [2] can be used as an approximate distance oracle of size $\tilde{\mathcal{O}}(n)$ to support k failures, with query time $\tilde{\mathcal{O}}(k\sqrt{n})$ simply by updating the graph k-times, each time with an additional failed vertex. As explained in Appendix A, no approximate dynamic distance oracle for directed planar graph is known to this date. Another question of interest, is whether there is an exact distance oracle with near linear space that can support any number of failures in $o(\sqrt{n})$ query time.

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A Approximative distance oracle with failed vertices

As we mentioned in Section 1, Abraham et al. [2] gave the best known fully dynamic $(1+\epsilon)$-distance oracle for undirected planar graphs. First we summarize their methods and then we explain why similar methods seem not to work in the directed setting.

Current Approach In order to decompose the graph into regions, they use a shortest-path separator for planar graphs [34].

Lemma 20. *Given a subgraph C of G, one can find in linear time an edge $(u,v) \in E(G)\backslash E(T)$ such that $C \setminus (T_u \cup T_v)$ is divided into two subgraphs A, B of at most $2|C|/3$ vertices such that no edge of G links a vertex of A to a vertex of B. Moreover, A and B lie on different faces of the plane graph $T_u \cup T_v \cup \{(u,v)\}$*

Using Lemma 20 recursively, they define a separator hierarchy tree $\mathcal{T}$ as follows. The tree $\mathcal{T}$ is binary, each node ν of $\mathcal{T}$ corresponds to an edge (u,v) of G induced by Lemma 20 applied on some subgraph C of G. The root of $\mathcal{T}$ is the edge (u,v) of Lemma 20 in the case $C = G$. The two children of (u,v) are then the two edges corresponding to the two path separators when applying Lemma 20 to the subgraphs A and B, The decomposition stops whenever we find (u,v) for C such that $C \subseteq T_u \cup T_v$, that is there is no more subgraphs A and B. Such a tree $\mathcal{T}$ has depth $O(\log n)$ and can be constructed in $O(n \log n)$ time.

Using the recursive decomposition they first show how to compute failure free distance labels. For each region $R \in \mathcal{T}$ and for each vertex $v \in R$ they store the distance $\mathrm{d}(v,p)$ where p is *portal* of v. The portals are vertices on the boundary of R chosen in a manner that for every vertex $t \in \partial R$, v has a portal at distance $\mathcal{O}(\epsilon d(v,t))$. Given a source vertex s, and a target vertex t, taking the edges of the labels of s and t and the edges connecting portals that belong to the same separator, gives us a relatively small graph that approximates the distance $d(s,t)$ by a factor of $1+\epsilon$. Running the standard Dijkstra algorithm retrieve the required distance.

In order to extend their labels to handle faults, they double the failure free labels in the following manner. The fault tolerant label of a vertex v contains:

1. the failure free label of v (as described above)
2. for each portal p in the label of v, the failure free label of p.

By doing so, at query time the label of a fault contains enough information to represent paths that does not go through the fault. Given a source vertex s, a target vertex t and a set F of failed vertices, they construct a graph H containing all of the safe edges in the labels of s, t and vertices of F. An edge (u, v) is safe if $\forall f \in F : d(u, f) + d(f, v) > d(u, v)$. Although the distances $d(u, f)$ and $d(f, v)$ are unknown, they are able to approximate them. Note that any edge that represents a path that goes through some failed vertex will not be safe. Therefore the graph H contains only valid paths. They show H contains a path of length at most $(1 + \epsilon)d(s, t, F)$ from s to t. One of the main complications, is that a vertex may belong to many regions and thus his label might become too large. To solve this issue, they show that each region contains at most $O(\log n)$ special vertices (*apices*) that need to be handled differently. The rest of the vertices belong to only $O(\log n)$ regions.

Directed Graphs We show that similar methods seem not to work in directed graphs. Specifically, reachability cannot be satisfied under the assumption that the label of a vertex contains connections to $o(\sqrt{n})$ vertices in the graph.

Following is a description of a family of directed planar graphs. Note that we only describe the shortest path tree with n as its parameter. Given an integer n that is a multiple of three and a small constant c, let r be the root vertex of the tree T with three child vertices. Let T_1 and T_2 be two trees that emerge from the children of r, and let s be the third child of r. T_1 contains a path P with $1/3n + c$ vertices where the degree of each vertex on P is two. Let w be the last vertex on P. T_2 contains two consecutive vertices u and v. v is of degree one (u is its only neighbour) and u is of degree $n/3 - 2c - 4$. Finally, we add edges from s to all vertices on P. Inspect Fig. 4 for an illustration.

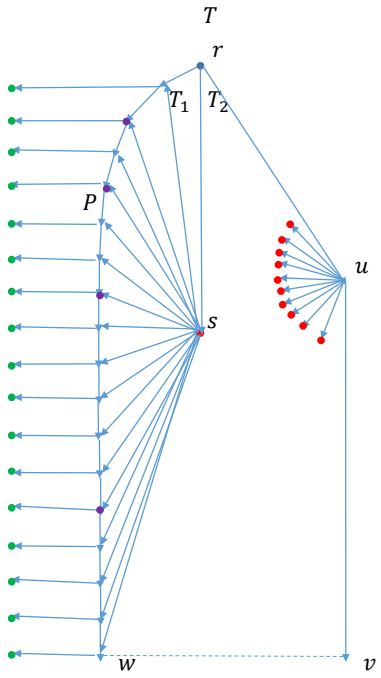

Figure 4: An example of the shortest path tree T where $n = 48$ and $c = 1$. Two paths emerge from r, T_1 and T_2. P has $1/3n + c = 17$ vertices and and u has a degree of $1/3n - 2c - 4 = 10$. The dotted line is one of the possible edges that satisfies Lemma 20. There are $1/3n - 2c - 3 = 11$ red vertices and $1/3n + c = 17$ green vertices which are on different sides of the separator. The purple vertices indicate a possible choice for portals of s.

Note that when $C = G$, one of the edges that satisfies Lemma 20 is (v, w) because $T_w \cup T_v \cup (v, w)$ divides the graph into two subgraphs of lengths $1/3n + c$ and $1/3n - 2c - 3$, both smaller than $2/3n$. Other choices for an edge would be an edge connecting v and a vertex relatively close to w on P. Any other edge would unbalance the division and thus not satisfy Lemma 20.

Given a portal labeling scheme such that each vertex keeps $\mathcal{O}(k)$ information, we choose the source vertex to be s. The label of s has distances to $\mathcal{O}(k)$ portals on P so there must be two portals $p_1, p_2 \in P$ with $\Omega(n/k)$ vertices between them that are not in the label of s. Setting any vertex f between p_1 and p_2 to be a failed vertex, s might lose reachability to the vertices between f and p_2, in contrast to the undirected case where p_2 could provide the relevant information. Let x be first vertex after f in P_1 that is reachable from s in $G \setminus \{f\}$. Unless the label of f stores information about the reachability of x from s, s would not be

able to reach x. If $n/k > k$ the label of f cannot store distances to all of the vertices between f and p_2. Concluding that if $k < \sqrt{n}$ for some vertex f between p_1 and p_2 there is such a vertex x that is not a portal of f hence not reachable from s using the labels of s and f. Finally we set the target vertex to be the neighbour of x that is not on P_1.

תקציר

אנו עוסקים בחישוב אוב שעונה על מרחקים קצרים בגרף מישורי ממושקל בהנתן קודקודים כושלים. יהי גרף G מישורי ממושקל, קודקוד מקור u, קודקוד יעד v ואוסף X של k קודקודים כושלים, אוב כזה מחזיר את המרחק הכי קצר בין u ל v אשר אינו עובר באף קודקוד של X. אנו מראים אוב אשר יכול להתמודד עם מספר לא מוגבל של כשולונות. בפרט, עבור k קבוע ועבור כל $q \in [1, n]$ אנו מראים אוב בגודל $\tilde{O}(\frac{n^{k+\frac{3}{2}}}{q^{2k+1}})$ אשר עונה לשאילתא בזמן q.

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למדעי המחשב
הבינתחומי
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המרכז הבינתחומי בהרצליה
בית-ספר אפי ארזי למדעי המחשב
התכנית לתואר שני (M.Sc.) - מסלול מחקרי

אוב מרחקים קצרים מדוייק לגרף מישורי עם קודודים כושלים

מאת
בנימין טבקה
בהנחיית ד"ר שי מוזס

עבודת תזה המוגשת כחלק מהדרישות לשם קבלת תואר מוסמך M.Sc.
במסלול המחקרי בבית ספר אפי ארזי למדעי המחשב, המרכז הבינתחומי הרצליה

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